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Refinements of T_1, compact and second countable topologies *

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Abstract

We prove that every T_1, compact and second countable topology has a refinement which is a topology of a Polish space and the Borel sets of the two topologies are the same. This shows immediately that every T_1, compact and second countable space has the standard Borel structure. A representation of all T_1, compact and second countable topologies as some special topologies on $P(\mathbb{N})$ is also given.

Key words: Borel sets; T_1 topological spaces; Compact spaces; Separation axioms

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We shall use the following notation. $\mathbb{N}$ stands for the set of all positive integers, $P(\mathbb{N})$ denotes the family of all subsets of $\mathbb{N}$. Assigning to each $A \in P(\mathbb{N})$ the sequence $(i_n: n \in \mathbb{N})$ such that $i_n = 1$ if $n \in A$ and $i_n = 0$ if $n \notin A$ we shall consider $P(\mathbb{N})$ as $\{0, 1\}^{\mathbb{N}}$ with the product topology. If $\mathscr{U}$ is any family of subsets of a set X, then by $\mathscr{O}(\mathscr{U})$ we shall denote the topology generated by the subbase $\mathscr{U}$. A *refinement* of a topology $\mathscr{O}$ is any topology richer than $\mathscr{O}$. For A being a subset of

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a product of sets by $\pi_n(A)$ we shall denote the projection of A onto the nth axis. The symbol $\operatorname{diag}(X^2)$ will stand for the set $\{(x, x): x \in X\}$. For a topological space $(X, \mathscr{O})$ and $A \subseteq X$ by $\mathscr{O} \mid A$ we shall denote the relative topology $\mathscr{O}$ on A.

A family $\mathscr{A}$ of sets is *compact in the sense of Marczewski* if for every $\mathscr{A}' \subseteq \mathscr{A}$ there is $\bigcap \mathscr{A}' \neq \emptyset$ whenever $\bigcap \mathscr{A}'' \neq \emptyset$ for all finite $\mathscr{A}'' \subseteq \mathscr{A}'$. If $\mathscr{A}$ is any family of sets then $\mathscr{A}_s$ will denote the family of all finite unions of elements of $\mathscr{A}$.

It is known that if A and B are Borel subsets of Polish spaces and A and B are of the same cardinality then the σ-algebras of Borel subsets of A and B are isomorphic. We say that a topological space X has a *standard Borel structure* if the σ-algebra of Borel subsets of X is isomorphic to the σ-algebra of Borel subsets of a Polish space (in [2] the term *standard Borel space* refers to a pair consisting of a set X and a σ-algebra of its subsets which contains X and which is isomorphic to the σ-algebra of Borel subsets of a Polish space, and X is not endowed with a topology).

We start with characterizing T_1 compact and second countable topological spaces. We shall extensively use the idea of Marczewski's characteristic function of sequence of sets (first used by Marczewski (Szpilrajn) in [6]).

Let $\mathscr{J}$ be an arbitrary nonempty family of finite subsets of $\mathbb{N}$ which is hereditary with respect to inclusion, i.e., satisfying

$$(J_1 \subseteq J_2 \text{ and } J_2 \in \mathscr{J}) \text{ implies } J_1 \in \mathscr{J}. \tag{1}$$

Now we define two families $\mathscr{G}$, $\mathscr{H} \subseteq P(\mathbb{N})$. The former consists of all sets the finite subsets of which are in $\mathscr{J}$, the latter consists of those elements of $\mathscr{G}$ which are maximal with respect to inclusion:

$$\mathscr{G} = \{G \in P(\mathbb{N}) : (\forall n \in \mathbb{N})[[1, n] \cap G \in \mathscr{J}]\} \tag{2}$$

and

$$\mathscr{H} = \{H \in \mathscr{G} : (\forall G \in \mathscr{G})[H \subseteq G \to H = G]\}. \tag{3}$$

Note that $\mathscr{G}$ is a closed subset of $P(\mathbb{N})$; in fact it is easy to observe that $\mathscr{G}$ is the closure of $\mathscr{J}$ (recall that $P(\mathbb{N})$ is identified with $\{0, 1\}^{\mathbb{N}}$ with the product topology).

Let us now consider the family

$$\boldsymbol{K} = \{\mathscr{K}_n : n \in \mathbb{N}\}, \tag{4}$$

where

$$\mathscr{K}_n = \{H \in \mathscr{H} : n \in H\}. \tag{5}$$

We shall show that $\boldsymbol{K}$ is compact in the sense of Marczewski. Let $\boldsymbol{K}' = \{\mathscr{K}_{n_1}, \mathscr{K}_{n_2}, \dots\} \subseteq \boldsymbol{K}$, and let us assume that $\mathscr{K}_{n_1} \cap \cdots \cap \mathscr{K}_{n_m} \neq \emptyset$, for each $m \in \mathbb{N}$. We have to show that $\bigcap \boldsymbol{K}' \neq \emptyset$. Pick an element $H_m \in \mathscr{K}_{n_1} \cap \cdots \cap \mathscr{K}_{n_m}$. As $\mathscr{G}$ is a closed subset of $P(\mathbb{N})$ we can find a subsequence $(H_{m_i})_i$ which is convergent to some $G \in \mathscr{G}$. Take any $H \in \mathscr{H}$ such that $G \subseteq H$. It is easy to observe that the set $\{n_1, n_2, \dots\}$ is contained in H and thus $H \in \bigcap \boldsymbol{K}'$. We have proved that $\boldsymbol{K}$ is compact in the sense of Marczewski, whence, as it is well known (see for instance [5, III, 1, the proof of Theorem 4]), $\boldsymbol{K}_s$ has the same property. Now we can consider $\mathscr{H}$ as a topological space taking $\boldsymbol{K}^{c}=\{\mathscr{H}\setminus\mathscr{K}\colon \mathscr{K}\in\boldsymbol{K}\}$ as a subbase. Obviously this space is second countable and compact. It is also T_1 because if H_1, $H_2\in\mathscr{H}$ and $H_1\neq H_2$, then as H_1 is maximal, there is $n\in H_1\setminus H_2$, whence $H_1\in\mathscr{K}_n$ and $H_2\notin\mathscr{K}_n$.

We shall prove now that the construction above describes all possibilities. More exactly, the following theorem holds.

Theorem 1. *Every T_1, compact and second countable topological space X is homeomorphic to $(\mathscr{H}, \mathscr{O}(\boldsymbol{K}^c))$ for some family $\mathscr{J}$ of finite subsets of $\mathbb{N}$ satisfying* (1), *where $\mathscr{H}$ and $\boldsymbol{K}$ are defined by* (2), (3), (4) *and* (5).

Proof. Let $\mathscr{U}$ be a countable base for X and let $\mathscr{C}=\{C_1, C_2,\dots\}=\mathscr{U}^c=\{X\setminus U\colon U\in\mathscr{U}\}$. Let $\mathscr{J}=\{J\colon J$ is a finite subset of $\mathbb{N}$ such that $\bigcap\{C_n\colon n\in J\}\neq\emptyset\}$. The families $\mathscr{G}$ and $\mathscr{H}$ are defined from $\mathscr{J}$ as previously (see (2) and (3)). Let $x\in X$ and $\phi(x)=\{n\colon x\in C_n\}$. We shall show now that ϕ is a homeomorphism between X and $\mathscr{H}$. We have $\bigcap\{C_n\colon n\in\phi(x)\}=\{x\}$ because X is T_1, and thus $\phi(x)\in\mathscr{H}$. Now let $H\in\mathscr{H}$. By the definition of $\mathscr{G}$ we have $\bigcap\{C_n\colon n\in A\}\neq\emptyset$ for any finite subset A of H, whence $\bigcap\{C_n\colon n\in H)\neq\emptyset$ because X is compact. If x_1, $x_2\in\bigcap\{C_n\colon n\in H\}$ and $x_1\neq x_2$, then there is $C_n\in\mathscr{C}$ such that $x_1\in C_n$ and $x_2\notin C_n$ because X is T_1. But then $H\cup\{n\}\in\mathscr{G}$ which is a contradiction with the definition of $\mathscr{H}$. Hence $\bigcap\{C_n\colon n\in H\}=\{x\}$ for some $x\in X$ and, obviously, $\phi(x)=H$. Thus ϕ is a $1-1$ mapping from X onto $\mathscr{H}$. To prove that ϕ is a homeomorphism it is enough to notice that $\phi(C_n)=\mathscr{K}_n$. □

We shall use the above characterization of T_1 compact and second countable topologies to prove that for each such topology there exists a finer one which is a topology of a Polish space generating the same Borel sets. Namely we shall prove the following theorem.

Theorem 2. *If $(X, \mathscr{O})$ is a T_1, compact and second countable topological space, then there is a topology $\mathscr{O}^*$ which is a refinement of $\mathscr{O}$ such that $(X, \mathscr{O}^*)$ is metrizable as a Polish space and the Borel sets generated by $\mathscr{O}^*$ are the same as those generated by $\mathscr{O}$.*

In the proof of the above theorem we shall use the following fact which we shall prove first.

Fact 3. *If X is a metrizable compact space, R is a binary relation on X and R is a closed subset of X^2 then the set of all elements maximal in R is a G_δ subset of X.*

Proof. It is enough to notice that

$$\{x\colon x \text{ is maximal with respect to } R\}=\left(\pi_1\left(R\setminus\operatorname{diag}(X^2)\right)\right)^{c}. \qquad\square$$

Proof of Theorem 2. In the view of Theorem 1 it is enough to prove the theorem for each space of the form $(\mathscr{H}, \mathscr{O}(\boldsymbol{K}^c))$ as in Theorem 1.

We have

$$\mathscr{K}_n = \{ A \in P(\mathbb{N}) : n \in A \} \cap \mathscr{H},$$

whence $\mathscr{K}_n$ is a clopen subset of $\mathscr{H}$ with the relative topology inherited from $P(\mathbb{N})$. Hence this topology extends that generated by a subbase $\boldsymbol{K}^c$.

The sets $\{A \in P(\mathbb{N}): n \in A\}$ are generators of the σ-algebra of Borel sets in $P(\mathbb{N})$ and $\mathscr{K}_n$ are closed sets in $\mathscr{H}$ with the topology $\mathscr{O}(\boldsymbol{K}^c)$. Thus $\mathscr{H}$ with the relative topology inherited from $P(\mathbb{N})$ has the same Borel sets as with the topology $\mathscr{O}(\boldsymbol{K}^c)$.

Thus the proof is complete because, as it follows from Fact 3, $\mathscr{H}$ is a G_δ subset of a Polish space $P(\mathbb{N})$ and thus is itself metrizable as a Polish space. □

The following corollary is immediate.

Corollary. *Every T_1, compact and second countable space has standard Borel structure.*

The following example due to Dow [3] shows that one cannot always obtain a Hausdorff and compact refinement of the T_1, compact and second countable topology.

Example 4 (Dow). Let a, $b \notin (0, 1]$, $a \neq b$, and

$$X = (0, 1] \cup \{a, b\}.$$

Let us define the neighbourhood base for a to consist of all sets of the form

$$U_n = \{a\} \cup \bigcup \left\{ \left((2i+2)^{-1}, (2i)^{-1} \right) : i \geqslant n \right\},$$

$n \in \mathbb{N}$, and the neighbourhood base for b to consist of all sets of the form

$$V_n = \{b\} \cup \bigcup \left\{ \left((2i+1)^{-1}, (2i-1)^{-1} \right) : i \geqslant n \right\},$$

$n \in \mathbb{N}$. The points of $(0, 1]$ have their usual neighbourhood bases. As any compact and Hausdorff refinement of this topology must remain the same on $(0, 1]$ it would be a two-point compactification of $(0, 1]$ and such compactification does not exist for the remainder of any compactification of $(0, 1]$ must be connected (see, for instance, [1, the first lines of the proof of Theorem 1]).

Below we give an example of a T_1, compact and second countable topology $\mathscr{S}$ that does not have any Hausdorff, σ-compact refinement consisting of Borel sets of $\mathscr{S}$. First we state the following fact.

Fact 5. *The space X described in Example 4 has the property that if K is its compact and dense subset, then $K = X$.*

Proof. It is easy to see that $K \cap (0, 1]$ must be dense in $(0, 1]$, and thus $(0, 1] \subseteq K$. If K did not contain either of the points a, b, then it would not be compact. The case where exactly one of the points a, b lies in K is also impossible. In fact if, for instance, $K = (0, 1] \cup \{a\}$, then the open cover of K consisting of the intervals $((2i+1)^{-1}, (2i-1)^{-1})$, $i \geqslant 1$, the set U_1 defined in Example 4 and the interval $(2^{-1}, 1]$ does not have any finite subcover, which contradicts the compactness of K. Thus we have $K = X$. □

Example 6. Let $(X, \mathscr{O})$ be a T_1, compact and second countable topological space having no Hausdorff and compact refinement. Assume also that $(X, \mathscr{O})$ has the property that each set compact and dense in X must be the whole X. The space described in Example 4 has all the properties required (see Fact 5).

Let the space $Y = X^{\omega}$ be equipped with the usual product topology $\mathscr{S}$. Of course, $(Y, \mathscr{S})$ is T_1, compact and second countable. We claim that this space does not have any Hausdorff, σ-compact refinement consisting of Borel sets of $\mathscr{S}$.

Let $\mathscr{O}^*$ be a refinement of $\mathscr{O}$ as in the conclusion of Theorem 2.

Let $\mathscr{P}$ be the topology on Y defined as a product of countably many copies of $\mathscr{O}^*$.

Assume now that $\mathscr{V}$ is a Hausdorff refinement of $\mathscr{S}$ consisting of Borel sets generated by $\mathscr{S}$. To prove that $\mathscr{V}$ cannot be σ-compact it is enough to show that every set $K \subseteq Y$ compact in the topology $\mathscr{V}$ is of the first category in the topology $\mathscr{P}$ (for $(Y, \mathscr{P})$ is metrizable as a Polish space).

Assume for a contradiction that K is of second category in the topology $\mathscr{P}$. The set K is closed in the topology $\mathscr{V}$, hence it is a Borel set in $\mathscr{P}$, thus possessing Baire's property in $\mathscr{P}$. By [4, Chapter 1, 11, IV, Corollary 2] there exist sets $U_1, \dots, U_n$ open in $\mathscr{O}^*$ such that the set $K \cap (U_1 \times \cdots \times U_n \times X \times X \times \cdots)$ is residual in the set $U_1 \times \cdots \times U_n \times X \times X \times \cdots$ (in the topology $\mathscr{P}$). By the Kuratowski–Ulam theorem [4, Chapter 2, 22, V, Corollary 1a] there is a sequence of points $c_1 \in U_1, \dots, c_n \in U_n$, $c_{n+2} \in X$, $c_{n+3} \in X, \dots$ such that the set

$$S = \{x \in X\colon (c_1, \dots, c_n, x, c_{n+2}, c_{n+3}, \dots) \in K\}$$

is residual and, therefore, dense in X with the topology $\mathscr{O}^*$. This implies that S is also dense in X in the topology $\mathscr{O}$, because $\mathscr{O}^*$ is a refinement of $\mathscr{O}$.

$$T = \{(c_1, \dots, c_n, x, c_{n+2}, c_{n+3}, \dots)\colon x \in X\}$$

is closed in X^{ω} with the topology $\mathscr{S}$, whence it is also closed in the topology $\mathscr{V}$. Hence the set $T \cap K$ is closed in K with the topology $\mathscr{V} \mid K$ and, because K is compact in this topology, $T \cap K$ is compact in the topology $\mathscr{V}$. Thus the set $T \cap K$ is also compact in the topology $\mathscr{S}$, whence its projection $S = \pi_{n+1}(T \cap K)$ is compact in the topology $\mathscr{O}$. So now we know that S is both dense and compact in X in the topology $\mathscr{O}$, whence $S = X$. Thus the projection of the topology $\mathscr{V} \mid T \cap K$ (note that $T \cap K = T$) onto the $n+1$ axis would be a Hausdorff and compact refinement of the topology $\mathscr{O}$, but such a refinement does not exist.

Remark. It follows from Theorem 2 that any T_1, compact and second countable topology has a refinement which is a topology of a G_δ subset of a Hausdorff, compact and second countable space. Using Example 6 we can show that this result cannot be improved, namely that there is a T_1, compact and second countable topology that does not have a refinement which is a topology of an F_σ subset of a Hausdorff, compact and second countable topological space. In particular, it will not have a Hausdorff, locally compact and second countable refinement.

The space $(Y, \mathscr{S})$ of Example 6 is just the one we need. Assume for a contradiction that it has a refinement which is the topology $\mathscr{T}$ of an F_σ subset of a Hausdorff, compact and second countable topological space. By Theorem 2 the space $(Y, \mathscr{S})$ has the standard Borel structure, namely that of the space $(Y, \mathscr{P})$ of Example 6. But then the identity function $\mathrm{id}: Y \to Y$, where the domain is equipped with the topology $\mathscr{P}$ and the range with the topology $\mathscr{T}$, is a Borel homeomorphism of Y [4, Chapter 3, 39, V, Theorem 1] and all Borel sets of $\mathscr{T}$ coincide with those of $\mathscr{P}$, and therefore of $\mathscr{S}$. Thus the refinement in question would consist of Borel sets of $\mathscr{S}$, but it does not exist as it was shown in Example 6.

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